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LETTER TO THE EDITOR

Lax pairs for ultra-discrete Painlevé cellular automata

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Abstract
Ultra-discrete versions of the discrete Painlevé equations are well known. However, evidence for their integrability has so far been restricted. In this letter, we show that their Lax pairs can be constructed and, furthermore, that compatibility conditions of the result yield the ultra-discrete Painlevé equation. For conciseness, we restrict our attention to a new d-P_{III}.

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1. Introduction

The discrete Painlevé equations are integrable discrete versions of the classical Painlevé equations that share many of the same properties [2, 3, 7]. We may apply what is known as the ultra-discretization method as shown in [10] to get versions of these equations [8] that are of interest because they can be interpreted as cellular automata. The method is believed to yield integrable equations. In this letter, we show for the first time how to construct Lax pairs for these automata.

For each variable (or parameter) v in a given equation, the ultra-discretization method requires that we introduce a new variable V defined by $v = e^{\frac{V}{\epsilon}}$. (Note that this places constraints on the variables involved.) The most important step is to then take the limit $\epsilon \to 0^+$ of the equation using the identity

$$\lim_{\epsilon\to 0^+} \epsilon \log\left(\sum_{i=1}^{n} e^{\frac{A_i}{\epsilon}}\right) = \max(A_i; i = 1, \ldots, n, 0). \tag{1.1}$$

In this letter, we examine the ultra-discrete version of the equation

$$y_{n+1}y_{n-1} = \frac{\alpha q^n + y_n^2}{1 + \alpha q^n y_n^2} \tag{1.2}$$

where $\alpha = (q-1)^2/q$ and $y_n = y(q^n)$. This is a discrete version of the third Painlevé equation and arises as the compatibility condition of a system of 2×2 linear problems (2.1) and (2.2) given in section 2. We chose to restrict our attention to this example because its Lax pair is simpler than those that have been given previously for other discrete versions of the third Painlevé equation (see, e.g., the 4×4 Lax pair given in section 5.2.5 of [3]).

To find the ultra-discrete version of equation (1.2), we introduce the ultra-discrete variables A, Q and Y_n given by

$$\alpha = \mathrm{e}^{\frac{A}{\epsilon}}, \qquad q = \mathrm{e}^{\frac{Q}{\epsilon}}, \qquad y_n = \mathrm{e}^{\frac{Y_n}{\epsilon}}.$$

By taking the limit as $\epsilon \to 0^+$, we obtain the ultra-discrete equation u-$\mathrm{P_{III}}$

$$Y_{n+1} + Y_{n-1} = \max(2Y_n, A + nQ) - \max(0, A + nQ + 2Y_n). \tag{1.3}$$

The restriction of α leads to $A = \max(Q, -Q) = |Q|$ for $Q \neq 0$. In section 2, we deduce the ultra-discrete Lax pair from that of equation (1.2). Most importantly, we show that the compatibility condition of the ultra-discrete Lax pair reduces exactly to equation (1.3).

The ultra-discrete equation can be restricted to the integers or to piecewise linear maps. Studies of ultra-discrete versions of many other equations such as the Lotka–Volterra equations in [4] and the first three Painlevé equations in [8] have now been made. The process has also been applied to the Modified Korteweg–de Vries equation. It was further shown that the ultra-discrete Modified Korteweg–de Vries equation admits a Lax pair [6]. What has not been studied is the whether ultra-discrete Painlevé equations admit Lax pairs and whether integrable cellular automata arises as a consequence of the Lax pair associated with a Painlevé equation. The main purpose of this letter is to demonstrate that this is in fact the case.

In section 2, we find the Lax pair of u-$\mathrm{P_{III}}$, i.e., equation (1.3), and show how to deduce the latter as a compatibility condition of the Lax pair. In section 3, we describe the qualitative behaviours of the solutions of u-$\mathrm{P_{III}}$. We end the letter with a conclusion in section 4.

2. Lax pair of u-$\mathrm{P_{III}}$

Equation (1.2) arises as the compatibility condition of the following Lax pair:

$$\phi(qx, k) = \begin{pmatrix} y_{n+1}/y_n & (q-1)k^2x/y_n \\ (q-1)xy_{n+1} & 1 \end{pmatrix} \phi(x, k) \tag{2.1}$$

$$\phi(x, qk) = \begin{pmatrix} y_{n+1}/y_n + \alpha x y_n y_{n+1} & (q-1)k^2x/y_n + \left(1 - \frac{1}{q}\right)y_n \\ \frac{(q-1)}{q^2}(k^2y_n)^{-1} + \left(1 - \frac{1}{q}\right)xy_n & \frac{1}{q}(y_n/y_{n+1} + \alpha x/(y_{n+1}y_n)) \end{pmatrix} \phi(x, k) \tag{2.2}$$

where $k = k_0q^m$ and $x = x_0q^n$. The first question we address in this section is what the ultra-discrete analogue of these equations are. The second question we address is how to consider the compatibility conditions of the resulting system.

Suppose $\phi(x, k) = \phi(q^n, q^m) = \binom{u_n^m}{v_n^m}$, then we may rewrite (2.1) and (2.2) as the following component equations:

$$u_{n+1}^m = \frac{y_{n+1}}{y_n}u_n^m + (\alpha q)^{1/2}q^{2m+n}y_nv_n^m \tag{2.3a}$$

$$v_{n+1}^m = (\alpha q)^{1/2}q^ny_{n+1}u_n^m + v_n^m \tag{2.3b}$$

$$u_n^{m+1} = \frac{y_{n+1}}{y_n}u_n^m + \alpha q^n y_n y_{n+1}u_n^m + \frac{(\alpha q)^{1/2}q^{2m+n}}{y_n}v_n^m + \left(\frac{\alpha}{q}\right)^{\frac{1}{2}}y_nv_n^m \tag{2.4a}$$

$$v_n^{m+1} = (\alpha q)^{1/2} q^{-(2m+2)} \frac{u_n^m}{y_n} + \left(\frac{\alpha}{q}\right)^{\frac{1}{2}} q^n y_n u_n^m + \frac{v_n^m}{q y_{n+1}} + \alpha \frac{q^{n-1} v_n^m}{y_{n+1} y_n} \tag{2.4b}$$

where we have used the fact that $(q - 1) = (\alpha q)^{\frac{1}{2}}$. (We assume below that α and q are positive real for simplicity.) A key observation in reducing this system of equations is that (2.4*a*) and (2.4*b*) may be expressed as linear combinations of both (2.3*a*) and (2.3*b*). From this observation we may write (2.4*a*) and (2.4*b*) as

$$u_n^{m+1} = u_{n+1}^m + \sqrt{\frac{\alpha}{q}} y_n v_{n+1}^m \tag{2.5a}$$

$$v_n^{m+1} = \frac{\sqrt{\alpha q}}{q^2 k^2 y_{n+1}} u_{n+1}^m + \frac{y_n}{q y_{n+1}} v_{n+1}^m. \tag{2.5b}$$

These imply that we can write (2.1) and (2.2) in the form

$$\phi(qx, k) = \begin{pmatrix} y_{n+1}/y_n & \sqrt{\alpha q} k^2 x / y_n \\ \sqrt{\alpha q} x y_{n+1} & 1 \end{pmatrix} \phi(x, k) \tag{2.6}$$

$$\phi(x, qk) = \begin{pmatrix} 1 & \sqrt{\frac{\alpha}{q}} y_n \\ \frac{\sqrt{\alpha q}}{q^2 k^2 y_{n+1}} & \frac{y_n}{y_{n+1} q} \end{pmatrix} \phi(qx, k). \tag{2.7}$$

Now we use the set of ultra-discrete variables $\alpha = e^{\frac{A}{\epsilon}}$, $q = e^{\frac{Q}{\epsilon}}$, $y_n = e^{\frac{Y_n}{\epsilon}}$, $u_n^m = e^{\frac{U_n^m}{\epsilon}}$ and $v_n^m = e^{\frac{V_n^m}{\epsilon}}$. By letting $\epsilon \to 0^+$ and using identity (1.1) we arrive at the set of ultra-discrete equations

$$U_{n+1}^m = \max\left(U_n^m + Y_{n+1} - Y_n, V_n^m + \frac{A}{2} + \left(2m + n + \frac{1}{2}\right) Q - Y_n\right) \tag{2.8a}$$

$$V_{n+1}^m = \max\left(U_n^m + \frac{A}{2} + \left(n + \frac{1}{2}\right) Q + Y_{n+1}, V_n^m\right) \tag{2.8b}$$

$$U_n^{m+1} = \max\left(U_{n+1}^m, V_{n+1}^m + \frac{A}{2} - \frac{Q}{2} + Y_n\right) \tag{2.9a}$$

$$V_n^{m+1} = \max\left(U_{n+1}^m + \frac{A}{2} - \left(2m + \frac{3}{2}\right) Q - Y_{n+1}, V_{n+1}^m + Y_n - Y_{n+1} - Q\right). \tag{2.9b}$$

To deduce the compatibility conditions for this system of linear equations, we introduce a notation, first used in [6]. We define the multiplication of two matrices A and B to be

$$[A \otimes B]_{ij} = \max_{1 \leqslant k \leqslant 2} (A_{ik} + B_{kj}). \tag{2.10}$$

We also define the multiplication of a matrix A by a vector v by

$$[A \otimes v]_i = \max_{1 \leqslant k \leqslant 2} (A_{ik} + v_k). \tag{2.11}$$

Thus, we may rewrite (2.8*a*)–(2.9*b*) as the pair of equations

$$\phi((n+1)Q, mQ) = \begin{pmatrix} Y_{n+1} - Y_n & \frac{A}{2} + \left(2m + n + \frac{1}{2}\right) Q - Y_n \\ \frac{A}{2} + \left(n + \frac{1}{2}\right) Q + Y_{n+1} & 0 \end{pmatrix} \otimes \phi(nQ, mQ) \tag{2.12}$$

$$\phi(nQ,(m+1)Q) = \begin{pmatrix} 0 & \frac{A}{2}-\frac{Q}{2}+Y_n \\ \frac{A}{2}-\left(2m+\frac{3}{2}\right)Q-Y_{n+1} & Y_n-Y_{n+1}-Q \end{pmatrix} \otimes \phi((n+1)Q,mQ). \tag{2.13}$$

We propose that this set of ultra-discrete equations is the Lax pair for the ultra-discrete equation ud-P_{III} in (1.3). For simplicity we write these equations as

$$\phi((n+1)Q,mQ) = L(nQ,mQ)\otimes\phi(nQ,mQ) \tag{2.14}$$

$$\phi(nQ,(m+1)Q) = M(nQ,mQ)\otimes\phi((n+1)Q,mQ). \tag{2.15}$$

Thus, by expressing $\phi((n+1)Q,(m+1)Q)$ in two different ways in terms of $\phi((n+1)Q,mQ)$ we see that the compatibility condition can be written as

$$L(nQ,(m+1)Q)\otimes M(nQ,mQ) = M((n+1)Q,mQ)\otimes L((n+1)Q,mQ). \tag{2.16}$$

By looking at $[L(nQ,(m+1)Q)\otimes M(nQ,mQ)]_{11}$ and $[M((n+1)Q,mQ)\otimes L((n+1)Q,mQ)]_{11}$ we arrive at the condition

$$\max(Y_{n+1}-Y_n, A+(n+1)Q-Y_n-Y_{n+1}) = \max(Y_{n+2}-Y_{n+1}, A+(n+1)Q+Y_{n+1}+Y_{n+2}).$$

Now we use the fact that $\max(a+b,c+b) = b+\max(a,c)$ to rewrite this equation as

$$-Y_{n+1}-Y_n+\max(2Y_{n+1}, A+(n+1)Q) = Y_{n+2}-Y_{n+1}+\max(0, A+(n+1)Q+2Y_{n+1}).$$

Clearly, this is equivalent to

$$Y_{n+2}+Y_n = \max(2Y_{n+1}, A+(n+1)Q) - \max(0, A+(n+1)Q+2Y_{n+1})$$

which is equation (1.3) with n replaced by $n+1$.

Similarly, by comparing the (2, 2) entries we get

$$\begin{aligned}&\max(A+nQ+Y_{n+1}+Y_n, Y_n-Y_{n+1}-Q)\\ &\quad= \max(A+nQ-Y_{n+2}-Y_{n+1}, Y_{n+1}-Y_{n+2}-Q)\end{aligned} \tag{2.17}$$

which may be rewritten as (1.3).

The (2, 1) entries, on the other hand, yield

$$\begin{aligned}&\max\left(\frac{A}{2}+\left(n+\frac{1}{2}\right)Q+Y_{n+1}, \frac{A}{2}-\left(2m+\frac{3}{2}\right)Q-Y_{n+1}\right)\\ &\quad= \max\left(\frac{A}{2}-\left(2m+\frac{3}{2}\right)Q-Y_{n+1}, \frac{A}{2}+\left(n+\frac{1}{2}\right)Q+Y_{n+1}\right).\end{aligned} \tag{2.18}$$

This is clearly an identity.

Similarly, by comparing the (1, 2) entries we have

$$\begin{aligned}&\max\left(\frac{A}{2}-\frac{Q}{2}+Y_{n+1}, \frac{A}{2}+\left(2m+n+\frac{3}{2}\right)Q-Y_{n+1}\right)\\ &\quad= \max\left(\frac{A}{2}+\left(2m+n+\frac{3}{2}\right)Q-Y_{n+1}, \frac{A}{2}-\frac{Q}{2}+Y_{n+1}\right)\end{aligned} \tag{2.19}$$

which is another identity. Thus, the compatibility of (2.12) and (2.13) is (1.3).

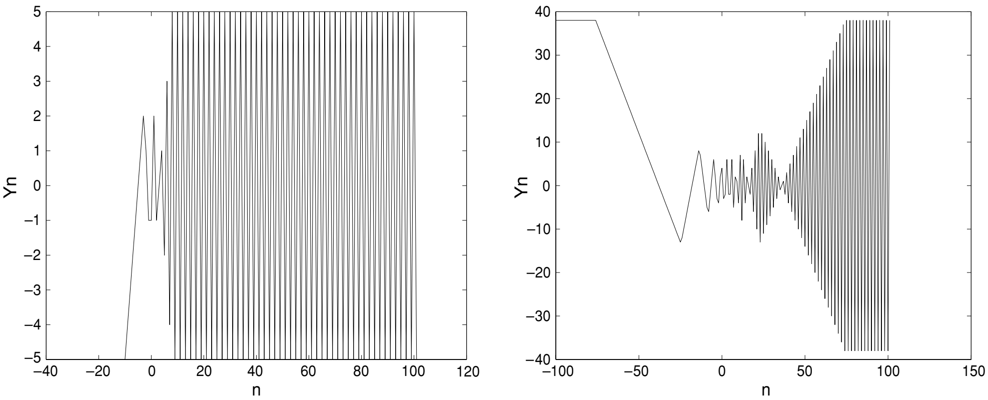

Figure 1. Solutions of (1.3) if $Y_0 = -1, Y_1 = -1$ and $Q = 1$ (left) and $Y_0 = 2, Y_1 = 4$ and $Q = 1$ (right). Both systems tend towards some form of periodic behaviour determined by the initial conditions.

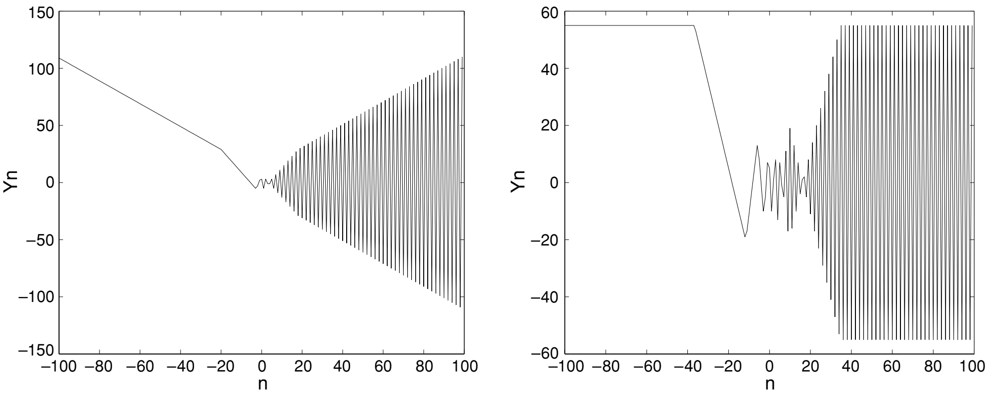

Figure 2. Solutions of (1.3) if $Y_0 = 3, Y_1 = 2$ and $Q = 3$ (left) and $Y_0 = 7, Y_1 = 5$ and $Q = 3$ (right). The initial conditions here have determined whether the system is convergent or divergent.

3. Solutions

When considering solutions of (1.3), we may restrict our attention to the case when $Q > 0$. The solutions are then completely determined by Y_0 and Y_1. We will also restrict our attention to the integers, in particular the cases where $\gcd(Q, Y_0, Y_1) = 1$. Figures 1–3 show a variety of distinct behavioural patterns observed for Y_n all determined by the initial conditions and Q.

According to the figures, we have a number of different behaviours for large n. Some of the classes of behaviour may be analysed, such as those in figures 1 and 2. To do this, we use the following notation, $Y_{2k+1} = U_k$ and $Y_{2k} = V_k$. Using these variables we write (1.3) as the following set of coupled equations:

$$U_k + U_{k-1} = \max(2V_k, A + 2kQ) - \max(0, A + 2kQ + 2V_k) \tag{3.1}$$

$$V_k + V_{k-1} = \max(2U_{k-1}, A + (2k-1)Q) - \max(0, A + (2k-1)Q + 2U_{k-1}). \tag{3.2}$$

From figure 1, we know that for large k, letting U_k be proportional to N^+ and V_k to be proportional to M^+, from (3.1) we have that for large k, $2M^+ = -2N^+$, and from (3.2),

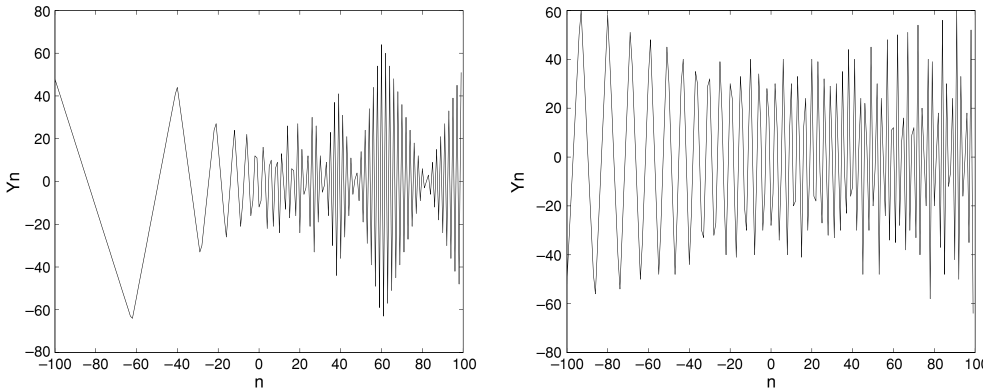

Figure 3. Solution of (1.3) if $Y_0 = 11$, $Y_1 = -12$ and $Q = 1$ (left) and $Y_0 = -16$, $Y_1 = 28$ and $Q = 1$ (right).

$2N^+ = -2M^+$. For large negative k, by letting U_k be proportional to M^- and V_k be proportional to N^-, from (3.1) we have for large negative k, $2M^- = 2N^-$ and from (3.2) we have $2N^- = 2M^-$. This implies that having Y_n alternate between M and $-M$ for large n is consistent with the equations. Also for large negative n, Y_n being constant is consistent with (1.3). Similarly, letting U_k be proportional to a^+k and V_k be proportional to b^+k, from (3.1) for large k we have $2ka^+ = 2k(\max(b^+, Q) - \max(0, (b^+ + Q)))$ and from (3.2) we have $2kb^+ = 2k(\max(a^+, Q) - \max(0, (a^+ + Q)))$, these two statements imply that $a^+ = -b^+ = \pm Q$ or $a^+ = -b^+$ where $|a^+| \leqslant Q$. This includes the 0 solution. Similarly, for large negative k letting U_k be proportional to a^-k and V_k be proportional to b^-k, from (3.1) we have for large negative k we have that $a^- = \min(b^-, Q) - \min(0, Q + b^-)$ and $b^- = \min(a^-, Q) - \min(0, Q + a^-)$. This admits the solution $a^- = b^- = \pm Q$ and also $a^- = b^-$ where $|a^-| \leqslant Q$. This means that alternating linear growth of Y_n at a rate of $\pm\frac{Q}{2}n$ for n positive is consistent with (1.3). Also linear growth at a rate of $\pm\frac{Q}{2}n$ is consistent with (1.3). Figure 3 shows other behaviour for Y_n is also observed.

4. Conclusion

Lax pairs of discrete or difference equations provide strong evidence of the integrability of such equations. They have been studied many times in the past [1]. Two more recent examples that consider partial difference equations and cellular automata can be found in [5, 9]. Whether the ultra-discretization of Lax pairs known for difference equations give Lax pairs for the ultra-discrete Painlevé equations has been unclear until now. In this letter, we provide evidence for the first time that this is the case.

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